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STIFFENING EFFECT OF UNCRACKED BRICKS IN THE STABILITY OF MASONRY PIERS

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Abstract—If the joints in a masonry structure are made of a no-tension, (or low tension) material and the bricks have a reasonable tensile strength the masonry will consist of alternative layers of cracked and uncracked material. Such a structure, when bent, will have a greater curvature than one made wholly of tension resisting material, but will display a smaller curvature as against a pier in bending where the cracked side of the structure lacks all strength and can be regarded as non-existing. The theoretical investigation will show how the curvature decreases as a result of stiffening effect of the uncracked layers of brick with a corresponding increase in the stability.

NOTATION

b width of pier
d depth of pier
e eccentricity at top of pier
t^2 $(\frac{1}{2}d-\delta)^{-1}$
x, v coordinates
x_c length of cracked region
z distance of line of action of load from concave face
A $2P/9Eb$
A_s $2P\bar{\rho}_m/9Eb$
B $\alpha^2\phi + 4(1-\phi/\beta)\bar{\rho}_m/3$
C $\ln(\sqrt{\beta/\phi}+\sqrt{\beta/\phi-1})$
E, E_j, E_b apparent elastic modulus of masonry structure, elastic modulus of joint, brick
L length of column
P axial load
α $1+|\sigma_t/\sigma_{av}|$
β $\frac{1}{2}-\alpha/6$
β_n $\beta^n/\phi-\beta^{n-1}$
ϕ $\frac{1}{2}-\delta/d$
δ eccentricity at bottom of pier
η_1, η_2 nondimensional thickness of joint, brick
$\bar{\rho}_m$ nondimensional mean equivalent curvature
ρ_1, ρ_2 curvature of joint, brick
σ_t, σ_{av} tensile resistance, P/bd

1. INTRODUCTION

Analytical investigations on the stability of masonry piers hinge on the fact that the effective depth, d, of the section decreases after cracking occurs and a cracked region forms in the pier. This region is separated from the uncracked one by the dotted line in Fig. 1. The depth of a crack is the variable $d-3z$ and disappears at x_c above the base. This problem has been considered by Angervo[1], Chapman and Slatford[2], Royen[3] and Frisch-Fay[4]. A common assumption made in these discussions is that for a no-tension material the stress distribution is trapezoidal in the uncracked zone following the stress law of a homogeneous section, and triangular in the cracked region (Fig. 1). If the material can resist a small amount of tension a tensile stress will build up on the convex face in the uncracked zone and will define the boundary between the partially cracked and the wholly uncracked sections.

As far as the partially cracked sections are concerned (i.e. the sections extending from $x=0$ to $x=x_c$) the assumption has been made in[1–4] that the cracked part of these sections is devoid of all strength and can be regarded as non existing material. A more realistic approach is to look at the cracked part as a combination of cracked and uncracked layers of material. As seen in Fig. 1 the brick and the horizontal joint, representing the cracked and uncracked layers, alternate and attract different types of curvatures. It will be assumed in this discussion that cracking will occur in the mortar layers (material 1) but not the brick layers (material 2).

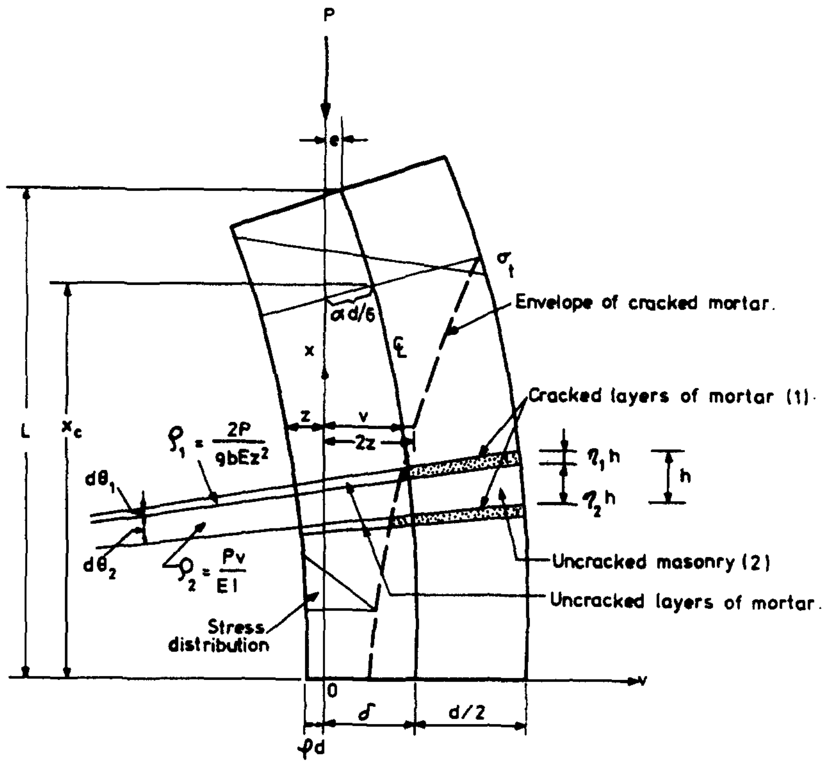

Fig. 1. Deflected shape of pier.

2. THE MODIFIED CURVATURE

The curvature in the cracked layer of mortar (Fig. 1) is[4]

$$\rho_1 = 2P/9Ebz^2 \tag{1}$$

while for the homogeneous layer of brick

$$\rho_2 = P(6 - 12z/d)/Ebd^2. \tag{2}$$

Equations (1) and (2) apply to a rectangular column of cross section $b \times d$ made of a material that can resist a small amount of tension $\sigma_t > 0$; the concentrated load P is eccentrically placed along the minor axis of the end section. The apparent modulus of elasticity of the combined masonry consisting of one joint plus one brick is E. From Sahlin[5],

$$E = (\eta_1/E_j + \eta_2/E_b)^{-1}.$$

If alternate layers 1 and 2 are arranged as in Fig. 1 a dimensionless equivalent curvature for the two layers can be represented as

$$\bar{\rho}_{eq} = \eta_1 + \eta_2\rho_2/\rho_1 = \eta_1 + 27\,\eta_2\frac{z^2}{d^2}(1 - 2z/d) \tag{3}$$

such that $\rho_{eq} = \bar{\rho}_{eq}\rho_1$.

Then,

$$\bar{\rho}_m = \frac{\int_0^{x_c} \bar{\rho}_{eq}\,dx}{x_c} \tag{4}$$

will be the mean equivalent curvature by which ρ_1 has to be modified to account for the stiffening effect of the uncracked bricks.

The curvature of the cracked section can now be had by changing eqn (1) to

$$\rho_1' = 2P\bar{\rho}_m/9Ebz^2. \tag{5}$$

Let $2P\bar{\rho}_m/9Eb = A\bar{\rho}_m = A_s$; a change of the variable in eqn (4) (see Appendix) leads to

$$\bar{\rho}_m = \frac{1}{x_c}\int_{z=\phi d}^{z=\beta d}\left(\eta_1 + 27\,\eta_2\frac{z^2}{d^2}(1-2z/d)\right)\frac{dz}{\sqrt{2A_s}\sqrt{(d/2-\delta)^{-1}-z^{-1}}}. \tag{4a}$$

After introducing A_s the following load-deflection relationship results (see Appendix):

$$L\sqrt{P/EI} = \sin^{-1}(\alpha\sqrt{\phi/B}) - \sin^{-1}\left(\frac{6e}{d}\sqrt{\phi/B}\right) + \sqrt{27/\bar{\rho}_m}\,[\sqrt{\beta\phi}\,\sqrt{\beta-\phi} + \sqrt{\phi^3}C] \tag{6}$$

where $\beta = 1/2 - \alpha/6$, $\phi = 1/2 - \delta/d$, $B = \alpha^2\phi + 4(1-\phi/\beta)\bar{\rho}_m/3$
and $C = \ln(\sqrt{\beta/\phi} + \sqrt{\beta/\phi - 1})$
The length of the craked region, x_c, is needed for eqn (4a). As shown in the Appendix

$$\bar{x}_c = x_c/L = 1 + \sqrt{EI/PL^2}\left[\sin^{-1}\left(\frac{6e}{d}\sqrt{\phi/B}\right) - \sin^{-1}(\alpha\sqrt{\phi/B})\right] \tag{7}$$

The simultaneous solution of eqns (4a), (6) and (7) will yield $\bar{x}_c$, $\bar{\rho}_m$ and PL^2/EI for various values of $\alpha, \delta/d$, e/d and η_1.

The variation of $\bar{\rho}_m$ vs δ/d for various values of α and e/d is plotted in Fig. 2. As expected, a low tension mortar leads to greater reduction of the average curvature than a no-tension joint.

The non-dimensional load vs deflection is plotted in Fig. 3. Here the variation of PL^2/EI vs δ/d is shown for $\alpha = 1$, 1.5, 2 and $e/d = 0.05$, 0.1, 0.15, 0.2 and 0.25. For comparison, the case of a no-tension joint ($\alpha = 1$) deriving no stiffening benefit from the brick is also shown in Fig. 3. For example, if $e/d = 0.05$, the stability of a pier made of no-tension joints increases by approx. 5% if the stiffening of the uncracked bricks is accounted for.

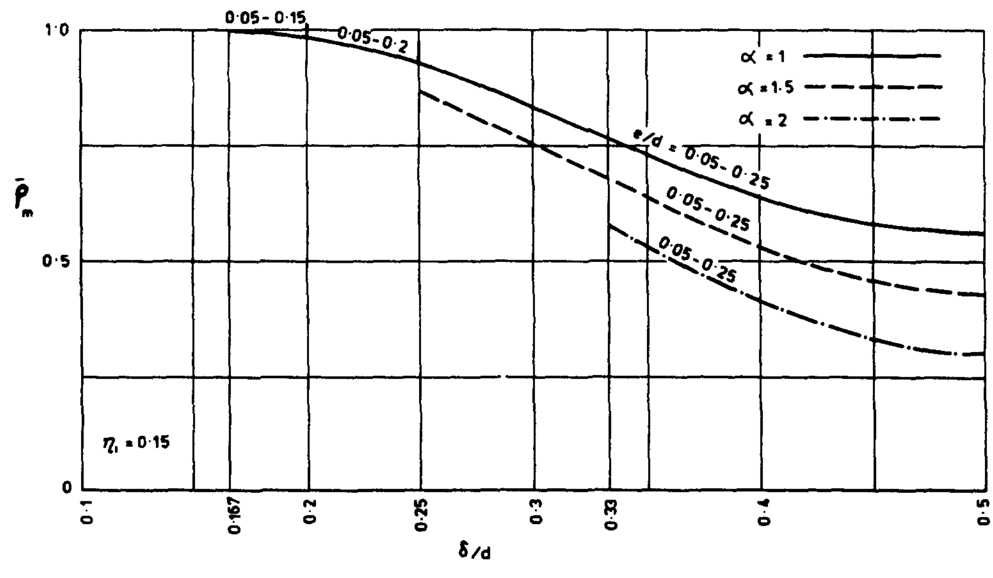

Fig. 2. Variation of $\bar{\rho}_m$ in terms of δ/d, e/d and α.

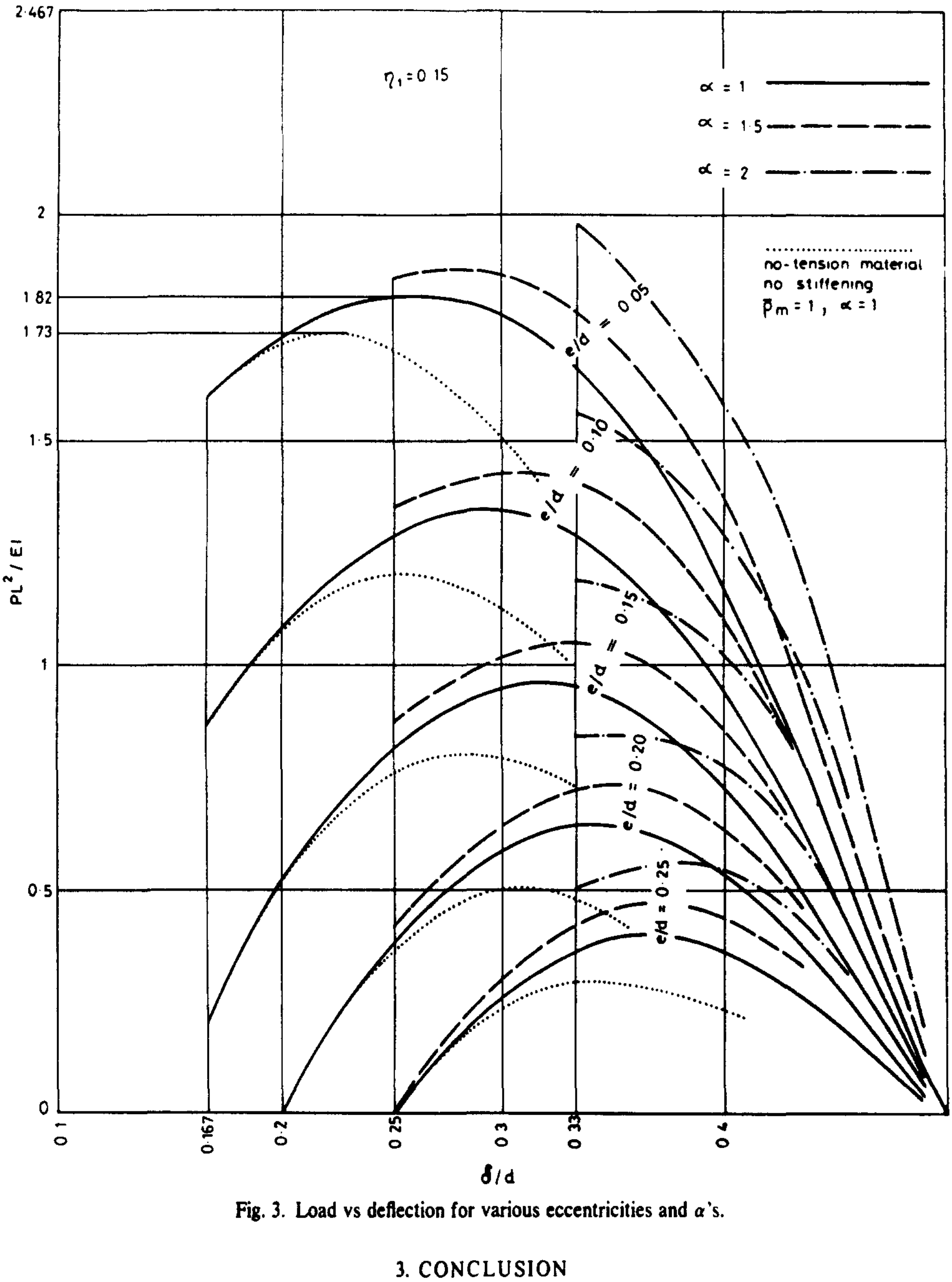

Fig. 3. Load vs deflection for various eccentricities and α's.

3. CONCLUSION

In replacing the idealized non-linear curvature of a partially cracked section by a curvature composed of a linear element and nonlinear element (corresponding to homogeneous and non-homogeneous sections, respectively) we can arrive at a more realistic equivalent curvature, one that is smaller than the less refined one. This approach leads to a modest increase in the elastic critical load of the pier.

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APPENDIX

The curvature in the cracked part is

$$\rho_1 = 2P/9Ebz^2 = A/z^2 \tag{1}$$

if the stiffening effect of the brick is ignored, and

$$\rho_1' = 2P\bar{\rho}_m/9Ebz^2 = A_s/z^2 \tag{5}$$

when such stiffening is included. Here

$$A = 2P/9Eb; \text{ and } A_s = 2P\bar{\rho}_m/9Eb.$$

Following the analysis in[4], the squares of the slopes in the cracked and uncracked parts, respectively, are

$$(dv/dx)^2 = 2A_s\left(\frac{1}{d/2-\delta} - \frac{1}{d/2-v}\right) \tag{8a}$$

$$(dv/dx)^2 = Pv^2/EI + C_1 \tag{8b}$$

$(dv/dx)|_{x=0} = 0$ having been satisfied. The modulus E refers to the masonry structure as a whole.

The two slopes in eqns (8a) and (8b) are identical at $v = \alpha d/6$ for at this section the cracked and uncracked parts meet and both have the same slope. It follows that

$$C_1 = \frac{P}{EI}\frac{\alpha^2 d^2}{36} + 2A_s\frac{1}{d/2-\delta} - 2A_s\frac{1}{d/2-\alpha d/6} \tag{9}$$

and

$$dv/dx = -\sqrt{\frac{3A}{2d}}\sqrt{\frac{4d\bar{\rho}_m/3}{d/2-\delta} + \alpha^2 - \frac{4\bar{\rho}_m}{3\beta} - \frac{36}{d^2}v^2} \tag{10}$$

for the uncracked region. Integration of eqn (10) and the satisfying of $x|_{v=e} = L$ results in

$$x\sqrt{54A/d^3} = L\sqrt{54A/d^3} + \sin^{-1}\left(\frac{6e}{d}\sqrt{\phi/B}\right) - \sin^{-1}\left(\frac{6v}{d}\sqrt{\phi/B}\right) \tag{11}$$

for the uncracked portion where

$$\phi = 1/2 - \delta/d \text{ and } B = \alpha^2\phi + 4\bar{\rho}_m(1-\phi/\beta)/3$$

For the cracked part (eqn 8a),

$$dx = -\frac{1}{\sqrt{2A_s}}\frac{dv}{\sqrt{\frac{1}{d/2-\delta} - \frac{1}{d/2-v}}}. \tag{12a}$$

With the substitution of

$$z = d/2 - v$$
$$t^2 = \frac{1}{d/2-\delta} = 1/\phi d \tag{12b}$$
and
$$w^2 = t^2 - 1/z$$

integration of eqn (12a) leads to the deflection of the cracked portion

$$x = \frac{1}{\sqrt{2A_s}}H(z) + C_2 \tag{13}$$

where

$$H(z) = \frac{1}{t^2}\sqrt{z}\sqrt{t^2 z - 1} + \frac{1}{t^3}\ln\left(t\sqrt{z} + \sqrt{t^2 z - 1}\right)$$

and

$$C_2 = L + \sqrt{\frac{d^3}{54A}}\left[\sin^{-1}\left(\frac{6e}{d}\sqrt{\phi/B}\right) - \sin^{-1}(\alpha\sqrt{\phi/B})\right] - \frac{1}{\sqrt{2A_s}}H(\beta d).$$

Here C_2 has been found by equating eqns (11) and (13) at $v = \alpha d/6$. There is from eqn (13)

$$\bar{x}_c = x_c/L = 1 + \sqrt{EI/PL^2}\left[\sin^{-1}\left(\frac{6e}{d}\sqrt{\phi/B}\right) - \sin^{-1}(\alpha\sqrt{\phi/B})\right]. \tag{14}$$

From $v|_{x=0} = \delta$ and eqn (13) we get the load-deflection relationship:

$$L\sqrt{P/EI} = \sin^{-1}(\alpha\sqrt{\phi/B}) - \sin^{-1}\left(\frac{6e}{d}\sqrt{\phi/B}\right) + \sqrt{27/\bar{\rho}_m}(\sqrt{\phi\beta}\sqrt{\beta-\phi} + \phi^{1.5}C) \tag{6}$$

If the changes in variables, outlined in eqns (12a) and (12b), are employed eqn (4a) will change to

$$\left.\begin{aligned} \bar{\rho}_m &= \frac{I_1 + I_2 - I_3}{L\bar{x}_c} \\ \text{where} \quad I_1 &= \frac{\eta_1}{\sqrt{2A_s}} \int_{z=\phi d}^{z=\beta d} \frac{\mathrm{d}z}{\sqrt{t^2 - 1/z}} \\ I_2 &= \eta_2 \frac{27}{d^2\sqrt{2A_s}} \int_{\phi d}^{\beta d} \frac{z^2\,\mathrm{d}z}{\sqrt{t^2 - 1/z}} \\ I_3 &= \eta_2 \frac{54}{d^3\sqrt{2A_s}} \int_{\phi d}^{\beta d} \frac{z^3\,\mathrm{d}z}{\sqrt{t^2 - 1/z}} \end{aligned}\right\} \tag{15}$$

Expanding eqns (15) results in

$$\left.\begin{aligned} I_1 &= \eta_1\sqrt{27/\bar{\rho}_m}\sqrt{EI/P}\,H_1 \\ I_2 &= 27\eta_2\sqrt{27/\bar{\rho}_m}\sqrt{EI/P}\,H_2 \\ I_3 &= 54\eta_2\sqrt{27/\bar{\rho}_m}\sqrt{EI/P}\,H_3 \\ \text{where} \quad & \\ H_1 &= \phi\sqrt{\beta_2} + \phi^{1.5}C \\ H_2 &= \frac{1}{3}\phi\sqrt{\beta_6} + \frac{5}{12}\phi^2\sqrt{\beta_4} + \frac{5}{8}\phi^3\sqrt{\beta_2} + \frac{5}{8}\phi^{3.5}C \\ H_3 &= \frac{1}{4}\phi\sqrt{\beta_8} + \frac{7}{24}\phi^2\sqrt{\beta_6} + \frac{35}{96}\phi^3\sqrt{\beta_4} + \frac{35}{64}\phi^4\sqrt{\beta_2} + \frac{35}{64}\phi^{4.5}C \end{aligned}\right\} \tag{16}$$

and

$$\beta_n = \beta^n/\phi - \beta^{n-1}.$$

After collecting these terms we finally arrive at the solution of eqn (4).

$$(\bar{\rho}_m)^{3/2} = \frac{\sqrt{27}}{\bar{x}_c}\sqrt{EI/PL^2}(\eta_1 H_1 + 27\eta_2(H_2 - 2H_3)). \tag{17}$$

The simultaneous solution of eqns (14), (6) and (17) will yield PL^2/EI, $\bar{\rho}_m$ and $\bar{x}_c$ provided that $\alpha/6 \leq \delta/d \leq 1/2$, $0 \leq \beta \leq 1/3$, $0 \leq \phi \leq \beta$ and $\alpha < 3$.